

ON AN INFINITE SERIES FOR $(1 + 1/x)^x$

CRISTINEL MORTICI AND YUE HU

ABSTRACT. The aim of this paper is to construct a new expansion of $(1 + 1/x)^x$ related to Carleman's inequality. Our results extend some results of Yang [Approximations for constant e and their applications J. Math. Anal. Appl. 262 (2001) 651-659].

1. INTRODUCTION

The following Carleman inequality [4]

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

whenever $a_n \geq 0$, $n = 1, 2, 3, \dots$, with $0 < \sum_{n=1}^{\infty} a_n < \infty$, has attracted the attention of many authors in the recent past. We refer here to the following results

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n+2}\right) a_n, \quad (\text{Bicheng and Debnath [2]})$$

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n+1/5}\right)^{-1/2} a_n, \quad (\text{Ping and Guozheng [6]})$$

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2cn + 4c/3 + 1/2}\right)^c a_n, \quad (\text{Yang [7]}).$$

Moreover, Yang [7] proved

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^6 \frac{b_k}{(n+1)^k}\right) a_n$$

with $b_1 = 1/2$, $b_2 = 1/24$, $b_3 = 1/48$, $b_4 = 73/5760$, $b_5 = 11/1280$, $b_6 = 1945/580608$, then conjectured that if

$$(1.1) \quad \left(1 + \frac{1}{x}\right)^x = e \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(x+1)^k}\right), \quad x > 0,$$

then $b_k > 0$, $k = 1, 2, 3, \dots$.

This open problem was recently solved by Yang [7], who proved

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(n+1)^k}\right) a_n,$$

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whenever $a_n \geq 0$, $n = 1, 2, 3, \dots$, with $0 < \sum_{n=1}^{\infty} a_n < \infty$, where $b_0 = 1$ and

$$b_n = \frac{1}{n} \left(- \sum_{k=0}^{n-2} \frac{b_{n-1-k}}{k+1} + \frac{1}{n+1} \right).$$

This conjecture was proved and discussed also by Yang [8], Gyllenberg and Yan [3], and Yue [9]. In the final part of his paper, Yang [8] remarked that in order to obtain better results, the right-hand side of (1.1) could be replaced by $e [1 - \sum_{n=1}^{\infty} (d_n / (x + \varepsilon)^n)]$, where $\varepsilon \in (0, 1]$ and $d_n = d_n(\varepsilon)$, but informations about values of ε are not provided. We prove in this paper that $\varepsilon = 11/12$ provides the fastest series $\sum_{n=1}^{\infty} (d_n / (x + \varepsilon)^n)$ and also formulas for coefficients d_n are given.

2. THE RESULTS

By truncation of the series

$$(2.1) \quad \left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{b_1}{n+1} - \frac{b_2}{(n+1)^2} - \dots\right)$$

we obtain approximations of any desired accuracy n^{-k} . The first approximation is

$$(2.2) \quad \left(1 + \frac{1}{n}\right)^n \approx e \left(1 - \frac{\frac{1}{2}}{n+1}\right)$$

but it is interesting to find the best approximation of the form

$$\left(1 + \frac{1}{n}\right)^n \approx e \left(1 - \frac{a}{n+b}\right), \quad \text{as } n \rightarrow \infty.$$

This problem was solved in [5], where the best values $a = 2$, $b = \frac{11}{6}$ were found. The proof of this fact is based on the following lemma, which is a powerful tool for measuring the speed of convergence.

Lemma 1. If $(\omega_n)_{n \geq 1}$ is convergent to zero and there exists the limit

$$\lim_{n \rightarrow \infty} n^k (\omega_n - \omega_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n \rightarrow \infty} n^{k-1} \omega_n = \frac{l}{k-1}.$$

Hence if replace $n+1$ by $n + \frac{11}{12}$ in (2.2), a better approximation can be obtained. An idea arises naturally: to construct a series (2.1) in negative powers of $n + \frac{11}{12}$.

This fact will also solve an open problem posed by Yang, who remarked in the final part of his paper [8] that in order to obtain better results, the right side of (2.1) could be replaced by $e [1 - \sum_{n=1}^{\infty} (d_n / (x + \varepsilon)^n)]$, where $\varepsilon \in (0, 1]$, but informations about values of ε are not provided.

Our above studies show that the value $\varepsilon = \frac{11}{12}$ gives indeed better results. The same method using Lemma 1 produces the series

$$(2.3) \quad \left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{\frac{1}{2}}{n + \frac{11}{12}} - \frac{\frac{5}{288}}{(n + \frac{11}{12})^3} - \frac{\frac{139}{17280}}{(n + \frac{11}{12})^4} - \frac{\frac{119}{23040}}{(n + \frac{11}{12})^5} - \dots\right)$$

which is better than (2.1), since by truncation after $k \geq 3$ terms of series (2.1), the last term is of order $n^{-(k-1)}$, while the last term of series (2.3) truncated after k terms is of order n^{-k} .

In order to obtain the next coefficient of $(n + \frac{11}{12})^{-2}$ in (2.3), we search the best approximation

$$\left(1 + \frac{1}{n}\right)^n \approx e \left(1 - \frac{\frac{1}{2}}{(n + \frac{11}{12})} - \frac{c}{(n + \frac{11}{12})^2}\right), \quad \text{as } n \rightarrow \infty.$$

Such an approximation is better as the relative error sequence defined by

$$\left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{\frac{1}{2}}{(n + \frac{11}{12})} - \frac{c}{(n + \frac{11}{12})^2}\right) \exp \omega_n, \quad n \geq 1,$$

converges faster to zero. By using computer algebra, we get

$$\omega_n - \omega_{n+1} = \frac{2c}{n^3} - \left(7c + \frac{5}{96}\right) \frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

According to Lemma 1, the fastest sequence ω_n is obtained for $c = 0$.

With $c = 0$, let us now search the best approximation of the form

$$\left(1 + \frac{1}{n}\right)^n \approx e \left(1 - \frac{\frac{1}{2}}{(n + \frac{11}{12})} - \frac{d}{(n + \frac{11}{12})^3}\right), \quad \text{as } n \rightarrow \infty.$$

For the corresponding relative error sequence w_n given by

$$\left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{\frac{1}{2}}{(n + \frac{11}{12})} - \frac{d}{(n + \frac{11}{12})^3}\right) \exp w_n, \quad n \geq 1,$$

we have

$$w_n - w_{n+1} = \left(3d - \frac{5}{96}\right) \frac{1}{n^4} + \left(-15d + \frac{493}{2160}\right) \frac{1}{n^5} + O\left(\frac{1}{n^6}\right).$$

The fastest sequence w_n is obtained when the coefficient of n^{-4} vanishes, that is $d = \frac{5}{288}$. More coefficients in (2.3) can be inductively obtained.

3. THE GENERAL TERM OF d_n

Now it is natural to ask the general term, or at least a recurrence relation of d_n in (2.3), that is

$$(3.1) \quad \left(1 + \frac{1}{n}\right)^n = e \left(1 - \sum_{k=1}^{\infty} \frac{d_k}{(n + \frac{11}{12})^k}\right).$$

By (2.3), we have $d_1 = \frac{1}{2}$, $d_2 = 0$, $d_3 = \frac{5}{288}$, $d_4 = \frac{139}{17280}$, $d_5 = \frac{119}{23040}$, \dots .

One idea for the complete characterization of d_n is to provide a formula in term of b_k , as we can see in the following

Theorem 1. *Let $b_0 = 1$ and $b_n = \frac{1}{n} \left(-\sum_{k=0}^{n-2} \frac{b_{n-1-k}}{k+1} + \frac{1}{n+1}\right)$, $n \geq 2$. Then if*

$$\left(1 + \frac{1}{m}\right)^m = e \left(d_0 - \frac{d_1}{m + \frac{11}{12}} - \frac{d_2}{(m + \frac{11}{12})^2} - \dots\right),$$

then $d_0 = 1$ and

$$d_s = \Gamma(s) \sum_{k=1}^s (-1)^{s-k} \frac{b_k}{\Gamma(s-k+1) \Gamma(k) 12^{s-k}}, \quad s = 1, 2, 3, \dots$$

Proof. First by the binomial formula, we have

$$\left(1 + \frac{1}{12}t\right)^{-k} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)12^n} t^n,$$

and with $t = \left(m + \frac{11}{12}\right)^{-1}$,

$$\left(1 - \frac{\frac{1}{12}}{m+1}\right)^k = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)12^n} \frac{1}{\left(m + \frac{11}{12}\right)^n}.$$

Now

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{b_k}{(m+1)^k} &= \sum_{k=0}^{\infty} \left(1 - \frac{\frac{1}{12}}{m+1}\right)^k \frac{b_k}{\left(m + \frac{11}{12}\right)^k} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n+1)12^n} \frac{b_k}{\left(m + \frac{11}{12}\right)^{n+k}} \\ &= \sum_{s=0}^{\infty} \frac{d_s}{\left(m + \frac{11}{12}\right)^s}, \end{aligned}$$

where

$$d_s = \sum_{k+n=s} (-1)^n \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n+1)12^n} b_k.$$

Now the conclusion follows by identifying the coefficients in

$$\begin{aligned} \left(1 + \frac{1}{m}\right)^m &= e \left(b_0 - \frac{b_1}{m+1} - \frac{b_2}{(m+1)^2} - \dots \right) \\ &= e \left(d_0 - \frac{d_1}{m + \frac{11}{12}} - \frac{d_2}{\left(m + \frac{11}{12}\right)^2} - \dots \right). \end{aligned}$$

□

We concentrate now to give a recurrence relation for sequence d_n . First we state the following

Lemma 2. *Let*

$$g(t) = \left(\frac{1 - \frac{11}{12}t}{1 + \frac{1}{12}t} \right)^{\frac{11}{12} - \frac{1}{t}}, \quad 0 < t < 1.$$

Then

$$(3.2) \quad g(t) = e(c_0 + c_1 t + c_2 t^2 + \dots),$$

where $c_0 = 1$ and

$$c_n = \frac{1}{n} \sum_{k=0}^{n-1} a_{n-k-1} c_k,$$

where

$$a_n = \frac{n+1}{12^{n+2}} \left(\frac{(-1)^{n+1} 11 - 11^{n+2}}{n+1} - \frac{(-1)^n - 11^{n+2}}{n+2} \right).$$

Proof. Using Maclaurin series, we have

$$\ln g(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{12^{n+1}} \left(\frac{(-1)^n 11 - 11^{n+1}}{n} - \frac{(-1)^{n-1} - 11^{n+1}}{n+1} \right) t^n,$$

thus

$$\frac{g'(t)}{g(t)} = \sum_{n=0}^{\infty} \frac{n+1}{12^{n+2}} \left(\frac{(-1)^{n+1} 11 - 11^{n+2}}{n+1} - \frac{(-1)^n - 11^{n+2}}{n+2} \right) t^n.$$

Now we can denote $g'(t) = g(t) \varphi(t)$, where

$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_n = \frac{n+1}{12^{n+2}} \left(\frac{(-1)^{n+1} 11 - 11^{n+2}}{n+1} - \frac{(-1)^n - 11^{n+2}}{n+2} \right).$$

Thanks to Leibniz rule,

$$g^{(n)}(t) = \sum_{k=0}^{n-1} \binom{n-1}{k} g^{(n-k-1)}(t) \varphi^{(k)}(t),$$

but $\varphi^{(k)}(0) = k!a_k$, so

$$g^{(n)}(0) = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!} a_k g^{(n-k-1)}(0).$$

Now

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!} a_k g^{(n-k-1)}(0) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} a_{n-k-1} \right) t^n \\ &= e \sum_{n=0}^{\infty} c_n t^n, \end{aligned}$$

where

$$c_n = \frac{1}{ne} \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} a_{n-k-1}$$

As

$$c_n = \frac{g^{(n)}(0)}{n!e}$$

we get

$$g^{(n)}(0) = (n-1)! \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} a_{n-k-1},$$

or

$$c_n = \frac{1}{n} \sum_{k=0}^{n-1} c_k a_{n-k-1},$$

which is the conclusion. \square

By taking $t = (m + \frac{11}{12})^{-1}$ in (3.2), we obtain the following

Theorem 2. *The following representation holds true*

$$\left(1 + \frac{1}{m}\right)^m = e \left(1 - \frac{d_1}{m + \frac{11}{12}} - \frac{d_2}{\left(m + \frac{11}{12}\right)^2} - \frac{d_3}{\left(m + \frac{11}{12}\right)^3} - \dots\right),$$

and

$$d_n = \frac{1}{n} \sum_{k=0}^{n-1} \frac{n-k}{12^{n-k+1}} \left(\frac{(-1)^{n-k} 11 - 11^{n-k+1}}{n-k} - \frac{(-1)^{n-k-1} - 11^{n-k+1}}{n-k+1} \right) d_k$$

(here $d_0 = -1$).

In the last part of this paper we give an integral representation of d_n . To do this, we make appeal to the following result stated in [1].

Lemma 3. *Let*

$$h(x) = (x+1) \left[e - \left(1 + \frac{1}{x}\right)^x \right] \quad (x > 0).$$

Then we have

$$h(x) = \frac{e}{2} + \frac{1}{\pi} \int_0^1 \frac{s^s (1-s)^{1-s} \sin(\pi s)}{x+s} ds.$$

Theorem 3. *Let d_n be the sequence defined by (2.3), and let*

$$g(s) = \frac{1}{\pi} s^s (1-s)^{1-s} \sin(\pi s).$$

Then

$$d_n = \frac{(-1)^n}{12^{n-1}} \left(-\frac{1}{2} + \frac{1}{e} \int_0^1 \frac{(12s-11)^{n-1} - 1}{s-1} g(s) ds \right), \quad n = 2, 3, \dots$$

Proof. By Lemma 3, we have

$$e - \left(1 + \frac{1}{x}\right)^x = \frac{e}{2} \frac{1}{1+x} + \int_0^1 \frac{g(s)}{(x+1)(x+s)} ds.$$

Thus, from Theorem 2, we have

$$e \left(\frac{d_1}{x + \frac{11}{12}} + \frac{d_2}{\left(x + \frac{11}{12}\right)^2} + \dots \right) = \frac{e}{2} \frac{1}{1+x} + \int_0^1 \frac{g(s)}{(x+1)(x+s)} ds.$$

With $t = \left(x + \frac{11}{12}\right)^{-1}$, we obtain

$$e (d_1 t + d_2 t^2 + \dots) = \frac{e}{2} \frac{12t}{12+t} + \int_0^1 \frac{144t^2}{12+t} \frac{g(s)}{12+12st-11t} ds.$$

Differentiation gives

$$d_n = \frac{(-1)^n}{12^{n-1}} \left(-\frac{1}{2} + \frac{1}{e} \int_0^1 \frac{(12s-11)^{n-1} - 1}{s-1} g(s) ds \right), \quad n = 2, 3, \dots$$

This completes the proof of Theorem 3. □

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VALAHIA UNIVERSITY OF TÂRGOVIȘTE, DEPARTMENT OF MATHEMATICS, BD. UNIRII 18, 130082
TÂRGOVIȘTE, ROMANIA

E-mail address: `cmortici@valahia.ro`

COLLEGE OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY,, JIAOZUO
CITY, HENAN 454000, P.R. CHINA.

E-mail address: `hhu3y2@163.com`